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On Interpolation; an Essay containing a simple Exposition of the Theory in its most useful practical applications, together with a general and complete Demonstration of the Methods of Quintisection of Briggs and Mouton for equal intervals; and of the process explained by Newton in his "Principia," for intervals of any magnitude whatever. By* M. FRÉD. MAURICE.

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THE calculation of extensive tables of logarithms could not fail to give rise to the method of interpolation; and such indeed was its origin. After Napier had made his memorable discovery, and he and Briggs had recognized the advantages that would result from the adoption of 10 as the base of a system of logarithms, it was Briggs who courageously undertook the immense labour required for calculating those famous tables, published in London in 1624, which have not been surpassed in extent and accuracy by any subsequent work. It is well known, however, that Briggs left a great *hiatus* in those tables, and that the logarithms of numbers between 20,000 and 90,000 are not to be found in them; but scarcely were the first tables printed, when Briggs undertook with fresh energy the computation of the Logarithms of the Trigonometrical Lines,

* From the *Connaissance des Temps* for the year 1847.

and he was on the point of terminating this vast enterprise when arrested by death. It was his friend Gellibrand who completed the work, and gave it to the world in 1633.

The great service rendered to science by this distinguished computer is not confined to the exact determination to a great number of decimal places of the logarithms calculated by him, and of their differences of several orders. Briggs has besides shown himself to be a mathematician of great skill, by his learned prefaces to the two great works to which we have referred. In these he has developed a host of ingenious methods invented by himself, with the view of combining rigorous exactness with facility of computation; and, with that object, he has given various remarkable methods which enabled him to verify his results to any degree of accuracy required.

One of the most remarkable of these is the singular process he explains (chapter xiii. of the Preface to his first tables, and chapter xii. of the Preface to his Trigonometrical Tables) for interpolating four intermediate values between each adjacent two of a series obtained by direct calculation, both the original and the interpolated values being separated by equal intervals. To this explanation he adds that, in publishing to 14 decimal places the logarithms of the 30,000 numbers comprised between 1 and 20,000, and between 90,000 and 100,000, it was his intention to give, by this process, an easy means of determining the 70,000 logarithms remaining to be calculated in order to complete those of the first hundred thousand natural numbers; and it may well be thought that he had already employed it to compute some of the logarithms contained in his valuable tables—we mean the Tables of Logarithms of Numbers, edited by himself in 1624.

This method, which is very remarkable, and preceded by nearly half a century the researches of Mouton and Regnauld upon a similar method of interpolation, is given by the author without demonstration. He only lays down the highly complicated rules of his method, and that in a manner wholly wanting in the clearness which, since the time of Euler, analysis has learnt how to throw on most subjects of which it treats. Furthermore, the process itself is more singular than easy of application: there seems nothing in it to guide the computer in choosing the limit of the corrections to which he must confine himself; and, moreover, the differences, of various orders, of the function under consideration, having reference to different values, might give rise to some confusion in practice.

To this defect in the explanation we must, no doubt, attribute the silence of mathematicians as to an invention so remarkable and of such merit. No one drew attention to its originality until, after the lapse of two centuries, Legendre made it the subject of a learned paper in the additions to the *Connaissance des Temps* for 1817. He there demonstrated, by means of a rigorous modern analysis, the reasons of the rules given by Briggs for deducing the differences to be employed in interpolation from those which he calls *mean differences*; showing that these last have, in fact, necessary relations with the differences of the original terms.

In that paper, the shortness of which is a matter of regret, Legendre confesses that his own demonstration is wanting in simplicity, and expresses a hope that some other proof may be found more closely akin to that which the author must undoubtedly have discovered, although he did not publish it. In the present paper we purpose to answer that appeal; and we believe we have succeeded, without using any process or method of investigation not known to mathematicians living in 1620, contemporary with Kepler and Harriot, and before Descartes and Fermat.

With this view, it appeared advisable, before giving our elementary demonstration of Briggs's curious process, to examine all the methods of investigation which were within his reach when about half way through his great work; and this will be the object of our first section. In the second, we shall describe, in Briggs's own words, that process which Legendre has deemed so worthy of attention; and in the third, it will be completely demonstrated, by a purely arithmetical method, whose only resemblance to modern analysis is due to the convenient notation so generally used at the present time.

We have contrasted with this rather obscure method the clear and convenient one published in 1670 by Mouton. He was a priest of Lyons, and the author of an astronomical work on the diameters of the sun and moon, containing several ideas which, considering the period in which he wrote, are very remarkable. But this method, which Mouton states was generalized by his friend and countryman Regnauld, is only described by him at some length, and is not demonstrated. Lalande was the first who undertook (in 1761) to demonstrate it for the three first orders of differences; and no further progress had been made, when Lagrange, in the Berlin *Memoirs* for 1792–3, published a demonstration as general as it was learned. Shortly after, Prony took this excellent method for the basis of the immense work executed under his direction, for

computing the great logarithmic tables called *Tables Du Cadastre* (Government survey), and, we are told, developed it in the preface in all its details. Those tables, however, which surpass both in extent and accuracy all that have been published, only exist in two manuscript copies, carefully preserved in different places, so that Prony's analysis has never been given to the public. It is true that Lacroix, in the third volume of his great treatise, has briefly given an outline of the method and of its demonstration; but as it is far from elementary, we trust that our readers will not consider as useless the more simple and complete exposition of it which we give in the fourth section. We shall there further show that Briggs's and Mouton's methods, notwithstanding their great dissimilarity, lead to exactly the same results, not only in their numerical applications, but likewise in the general formulæ. As, moreover, we have not omitted to give such details as may be practically useful in interpolating, it will be seen that the clear and easy process of Mouton offers unusual advantages from the symmetry of all its operations.

Hitherto we have only considered the case when the intervals between the given values are equal; but other cases occur in practice. For example, in some astronomical observations it happens that the times between the successive observations are unequal; and then, taking the time as the variable, the values given by observation are not equidistant, and the problem is to find from the given data the value of the observed element at any given time.

Newton gave the first solution of this problem in a lemma in his *Principia*, which, according to his custom, he did not stop to demonstrate; and Laplace, who adopted that solution in his method of calculating the orbit of a comet, also confined himself to its simple enunciation. We devote our fifth and last section to the general consideration of cases of this kind; and after giving a solution based on the same purely algebraical reasoning we have used throughout, we shall conclude with an easy demonstration of Newton's solution, under the form given to it by Laplace in his *Mécanique Céleste.*

Lastly, in an Appendix, we have considered more particularly the formulæ most frequently used in astronomical interpolations; and we have deduced them also from the same simple principles. We would particularly draw attention to the manner in which we there treat a formula which Stirling, in his excellent work on interpolation, had only obtained by induction, and to the additions we have given to M. Bessel's demonstration of that useful formula. The combined simplicity and clearness of all these calculations will, it is hoped, secure for them a careful perusal.

§ I.

1. Some persons who have carefully examined the prefaces of Briggs, have seen therein reason for thinking that he had anticipated Pascal, Wallis and Newton, in the discovery of the celebrated Binomial Theorem. Dr. Charles Hutton, for instance, in the introduction to his Tables, states this explicitly, and endeavours to explain how the fact may be reconciled with Newton's acknowledged claim to that most valuable discovery. We shall see, however, that Briggs may have arrived at a very analogous formula by a widely different course of reasoning, which would not necessarily require him to consider the raising of a binomial to a given power, either integral or fractional. We shall do our author full justice by allowing that he was led by the very nature of his investigations to find general formulæ showing the relations between the equidistant terms of any given series and their differences of any order. It is evident that the logarithms of successive numbers are quantities of this nature.

2. Thus, if we have a series of given quantities

$$u_0,\ u_1,\ u_2,\ u_3,\ \ldots\ldots,\ u_x,\ \ldots\ldots,$$

corresponding to values of the variable, x,

$$0,\ 1,\ 2,\ 3,\ \ldots\ldots\ x,\ \ldots\ldots,$$

it is easily seen that we may assume the general term of the series to be given by the formula

$$u_x=u_0+\mathrm{A}x+\mathrm{B}.x(x-1)+\mathrm{C}.x(x-1)(x-2)+\mathrm{D}.x(x-1)(x-2)(x-3)+\ldots$$

where the coefficients A, B, C, D, ... are to be determined by means of the conditions (obtained by putting x equal to 1, 2, 3, ... successively)

$$u_1=u_0+\mathrm{A},\qquad u_3=u_0+3\mathrm{A}+6\mathrm{B}+6\mathrm{C},$$
$$u_2=u_0+2\mathrm{A}+2\mathrm{B},\qquad u_4=u_0+4\mathrm{A}+12\mathrm{B}+24\mathrm{C}+24\mathrm{D},\ \ldots.,$$

Hence we deduce, representing by the symbol δ the difference of two of the quantities, and by δ^n, generally, a difference of the nth order,

$$\mathrm{A}=u_1-u_0=\delta u_0;$$

$$\mathrm{B}=\frac{u_2-2u_1+u_0}{2}=\frac{u_2-u_1-(u_1-u_0)}{2}=\frac{\delta(u_1-u_0)}{2}=\frac{\delta^2 u_0}{1.2};$$

$$\mathrm{C}=\frac{u_3-3u_2+3u_1-u_0}{1.2.3}=\frac{\delta^3 u_0}{1.2.3};$$

$$\mathrm{D}=\frac{u_4-4u_3+6u_2-4u_1+u_0}{1.2.3.4}=\frac{\delta^4 u_0}{1.2.3.4}.$$

Substituting these values, the formula for the general term becomes

$$(\mathrm{E}) \qquad u_x = u_0 + \frac{x}{1} \cdot \delta u_0 + \frac{x(x-1)}{1.2} \cdot \delta^2 u_0 + \frac{x(x-1)(x-2)}{1.2.3} \cdot \delta^3 u_0$$
$$+ \frac{x(x-1)(x-2)(x-3)}{1.2.3.4} \cdot \delta^4 u_0 + \dots.$$

3. This expression calls for two remarks:—

The first is, that the series may come to an end, either because x is a finite integer, and then all the coefficients after $\frac{x(x-1)\dots(x-\overline{x-1})}{1.2\dots x}$ will of necessity vanish; or because we meet with constant differences of the nth order, in which case all the terms after $\frac{x(x-1)\dots(x-n+1)}{1.2\dots n} \cdot \delta^n u_0$ will disappear, because $\delta^{n+1} u_0 = 0$, &c.

The second remark is, that we have not proved that the form which we have seen to prevail in the case of a few terms, will be generally true whatever the number of terms. But a very simple consideration will satisfy us on this point. Suppose that x is increased by unity; we shall have

$$u_{x+1} = u_x + \delta u_x;$$

$$\therefore \quad u_{x+1} = u_0 + \frac{x}{1} \cdot \delta u_0 + \frac{x(x-1)}{1.2} \cdot \delta^2 u_0 + \frac{x(x-1)(x-2)}{1.2.3} \cdot \delta^3 u_0 + \dots$$
$$+ 1.\delta u_0 + \frac{x}{1} \cdot \delta^2 u_0 \qquad + \frac{x(x-1)}{1.2} \cdot \delta^3 u_0 \qquad + \dots$$
$$= u_0 + \frac{x+1}{1} \cdot \delta u_0 + \frac{(x+1)x}{1.2} \cdot \delta^2 u_0 + \frac{(x+1)x(x-1)}{1.2.3} \cdot \delta^3 u_0 + \dots$$

Thus u_{x+1} is expressed by a series in perfect conformity with that of the formula (E); from which it may be strictly concluded that if the formula is proved for any term, u_x, it is true also for the next term, u_{x+1}; and by successive inductions it is true for all positive and integral values of x.

This formula (E) is so related to that of the binomial theorem, that a century and a half after Briggs's time it was at last written in the abridged form

$$u_x = \{1 + \delta\}^x u_0.$$

In fact, the formula for u_x results rigorously from the product of u_0 by the expansion of the xth power of the binomial $(1 + \delta)$.

4. We are able to see also, when the coefficients A, B, C, D, … have been determined, that the successive differences of u are expressed in series containing the successive values of u itself, with alternate signs and coefficients similar to those of the series

(E); so that, for example, up to the nth order inclusive, we have obtained the expression

$$\text{(D)} \quad \delta^n u_0 = u_n - \frac{n}{1} \cdot u_{n-1} + \frac{n(n-1)}{1.2} \cdot u_{n-2} - \frac{n(n-1)(n-2)}{1.2.3} \cdot u_{n-3} + \ldots$$

A line of reasoning similar to that of the preceding article will prove that this general form is also true when n becomes $n+1$.

In fact, $\delta^{n+1} u_0 = \delta^n (u_1 - u_0)$. But the value of $\delta^n u_1$ will be deduced from that of $\delta^n u_0$ just found, by writing u_{n+1} for u_n, u_n for u_{n-1}, &c. We shall then have

$$\begin{aligned} \delta^{n+1} u_0 &= u_{n+1} - \frac{n}{1} \cdot u_n + \frac{n(n-1)}{1.2} \cdot u_{n-1} - \frac{n(n-1)(n-2)}{1.2.3} \cdot u_{n-2} + \ldots . \\ &\qquad - 1.u_n + \frac{n}{1} \cdot u_{n-1} \qquad - \frac{n(n-1)}{1.2} \cdot u_{n-2} \qquad + \ldots . \\ &= u_{n+1} - \frac{n+1}{1} u_n + \frac{(n+1)n}{1.2} \cdot u_{n-1} - \frac{(n+1)n(n-1)}{1.2.3} \cdot u_{n-2} + \ldots \end{aligned}$$

But this expression being of exactly the same form as (D), we see that if that formula is true for the nth difference, it is also true for the $(n+1)$th difference; and by successive inductions it is true for any order of differences whatever.

This new formula received, about the same time as the previous one, the following concise and convenient form—

$$\delta^n u = (u-1)^n,$$

where care must be taken, in the expansion of the second member, to write throughout u_n instead of u^n, u_{n-1} instead of u^{n-1}, &c.; and in the last term, instead of 1 or u^0, to write u_0.

5. We have considered, (art. 2) a series of given quantities, $u_0, u_1, u_2, \ldots, u_x, \ldots$ or values of the function u_x, corresponding to the values of the variable $0, 1, 2, \ldots, x, \ldots$; and we have obtained the formula (E) for the general term u_x.

Let us now consider the series $u_0, u_h, u_{2h}, \ldots, u_z, \ldots$ where h is an integer, and where we must have $z = xh$, x being also an integer.

Then it is evident that the formula for the general term u_z will be deduced from that for u_x or from (E) by substituting for x in the second member its value $\frac{z}{h}$; for x takes the values $0, 1, 2, \ldots x, \ldots$ when z is according to our hypothesis made equal to $0, h, 2h, \ldots xh, \ldots$ successively; only it will be necessary to put Δu_0 or $u_h - u_0$, and its successive differences $\Delta^2 u_0$, $\Delta^3 u_0, \ldots$ in the place of δu_0, or $u_1 - u_0$, and its successive differences $\delta^2 u_0$, $\delta^3 u_0, \ldots$

Making these changes, the formula becomes

$$\text{(F)}\quad u_z = u_0 + \frac{z}{h}\Delta u_0 + \frac{z(z-h)}{1.2.h^2}\Delta^2 u_0 + \frac{z(z-h)(z-2h)}{1.2.3h^3}\Delta^3 u_0 + \ldots.$$

which may also be obtained directly, in just the same manner as the formula (E) was first obtained (art. 2), if we represent by Δu_0 the difference $u_h - u_0$.

6. This formula (F), which for values of z equal to 0, h, $2h$, . . . gives

$$\text{(Z)}\quad \begin{cases} u_0 = u_0 \\ u_h = u_0 + \Delta u_0 \\ u_{2h} = u_0 + 2\Delta u_0 + \Delta^2 u_0 \\ \&\text{c.} = \quad \&\text{c.} \end{cases}$$

will of necessity coincide with that deduced from the formula (E), by making x equal to 0, h, $2h$, . . . successively. Conversely, (E) must also coincide with (F), when z is made equal to 0, 1, 2, . . . successively, which gives

$$\text{(Z)}'\quad \begin{cases} u_0 = u_0 \\ u_1 = u_0 + \frac{1}{h}\Delta u_0 + \frac{1(1-h)}{1.2.h^2}\Delta^2 u_0 + \frac{1(1-h)(1-2h)}{1.2.3.h^3}\Delta^3 u_0 + \ldots. \\ u_2 = u_0 + \frac{2}{h}\Delta u_0 + \frac{2(2-h)}{1.2.h^2}\Delta^2 u_0 + \frac{2(2-h)(2-2h)}{1.2.3.h^3}\Delta^3 u_0 + \ldots. \end{cases}$$
$$\&\text{c.} = \quad \&\text{c.}$$

We thus conclude that the use of the single formula (F) will give the values of u_z for all integral values of z. Consequently, if we have computed a series of values of a function u_z, such as are shown in (Z), we see from (Z)$'$ that we can determine, by means of the known quantities, u_0, Δu_0, $\Delta^2 u_0 \ldots . \Delta^{x-1} u_0$, the $h-1$ values of u_z comprised between u_0 and u_h, or between u_h and u_{2h}, &c., and this is called *interpolating*; *i.e.*, if we have a series of values of u_z corresponding to values of z, that differ by the constant quantity h, we know how to insert, between any two adjacent values, $(h-1)$ intermediate values, equidistant like the first.

7. Similarly if the original successive values of z had been 0, 1, 2, 3, . . ., instead of 0, h, $2h$, . . . we should have had a formula exactly similar to (E), and which would be written thus:—

$$\text{(E}'\text{)}\quad u_z = u_0 + \frac{z}{1}\delta u_0 + \frac{z(z-1)}{1.2}\delta^2 u_0 + \frac{z(z-1)(z-2)}{1.2.3}\delta^3 u_0 + \ldots.$$

and the known values of u_z would be

$$\begin{aligned} u_0 &= u_0 \\ u_1 &= u_0 + \delta u_0 \\ u_2 &= u_0 + 2\delta u_0 + \delta^2 u_0 \\ \&\text{c.} &= \quad \&\text{c.} \end{aligned}$$

But if we now suppose z to have the values $0, \frac{1}{h}, \frac{2}{h}, \ldots \frac{x}{h}, \ldots$ successively, it will be evidently sufficient to put, for z, in (E′) its general value $\frac{x}{h}$; for (E′) is really the same formula as (E): they are both dependent upon increments of the independent variable equal to unity; so that in the form which (E′) then takes, increments of x equal to unity will give the successive values of u_z.

We shall have then

$$(\text{F}') \quad u_z = u_{\frac{x}{h}} = u_0 + \frac{x}{h}\delta u_0 + \frac{x(x-h)}{1.2.h^2}\delta^2 u_0 + \frac{x(x-h)(x-2h)}{1.2.3.h^3}\delta^3 u_0 + \ldots .$$

a formula, which, making x equal to $0, h, 2h, \ldots$, will give again the preceding values of $u_0, u_1, u_2, \ldots$, namely,

$$u_0, \ u_0+\delta u_0, \ u_0+2\delta u_0+\delta^2 u_0, \ \ldots .$$

Consequently, if we have a table showing, like the formula (E) the values of u_x for values of x equal to $0, 1, 2, 3, \ldots$, so that we know the values of $u_0, \delta u_0, \delta^2 u_0, \delta^3 u_0, \ldots$, the formula (F′) will give the $(h-1)$ values $u_{\frac{1}{h}}, u_{\frac{2}{h}}, \ldots u_{\frac{h-1}{h}}$, intermediate between u_0 and $u_{\frac{h}{h}}$ or u_1, or enable us to *interpolate* between u_0 and u_1 these $(h-1)$ values which are, like the original values, equidistant. It is evident that we may do the same for the second interval comprised between $u_{\frac{h}{h}}$ or u_1, and $u_{\frac{2h}{h}}$ or u_2; for the third, and for all subsequent intervals.

8. Such are the means of calculation which we may suppose Briggs to have possessed when he began to work at his great tables. In these operations we find really nothing beyond the general arithmetic then known; as far as theory goes, they only introduce the simple and precise ideas of the differences of various orders of given quantities, and the reciprocal relations between such quantities and differences.

We shall, however, soon see that these formulæ and ideas were more than sufficient to lead him to the singular method of interpolation which we are about to explain, and which we shall first present as it appears in the 13th chapter of the Preface to the *Arithmetica Logarithmica*, published at London in 1624.

9. We have said that these means were *more than sufficient* to lead Briggs to his method. We shall see in fact that it is highly probable that he never obtained the formulæ (F) and (F′); at all events, they are not required in order to obtain his results. More-

over, it may be remarked, that if he had only had before him the second of the formulæ (Z′) in art. 6, which gives

$$\delta u_0 = \frac{1}{h}\Delta u_0 + \frac{1(1-h)}{1.2.h^2}\Delta^2 u_0 + \dots$$

he would have been led to think that $\delta^n u_0$ must depend upon $\Delta^n u_0$, $\Delta^{n+1} u_0$, . . . ; and that, consequently, if $\Delta^n u_0$ were a *constant* difference, it was unnecessary to notice $\delta^{n+1} u_0$, $\delta^{n+2} u_0$, ... But, as will be seen in articles 11, 18 and 28, Briggs carries the formula for $\delta^4 u_0$ up to the *ten thousandth part* of $\delta^{20} u_0$, . . . , whilst, supposing $\Delta^5 u_0$ constant, he should only have taken $\delta^5 u_0$ into consideration, and should have neglected all differences of a higher order.

But these observations will arise more naturally as we proceed. Let us now hear Briggs's own description of his curious process.

§ II.

10. In the 13th chapter of the Preface to his Tables, Briggs wished to give the means of finding the logarithms which he had not calculated, namely, those of the numbers between 20,000 and 90,000; and for that purpose he proposed the following problem:— "Given a series of equidistant numbers and their logarithms, to find the logarithms of the four numbers interpolated at equal intervals between each adjacent two of the given numbers."

Thus, for example, Briggs having calculated the logarithms of all the numbers below 20,000, easily obtained, by adding the logarithm of 5 to those of the numbers 4,001, 4,002, 4,003, . . . , the logarithms of the numbers 20,005, 20,010, 20,015, . . . ; but he had still to find a means of obtaining, without too much labour, the logarithms of the four whole numbers comprised between each two of the last numbers.

But let the author speak for himself. "Take," he goes on to say, "the first, second, third, fourth, &c., differences of the logarithms of 2,115, 2,120, 2,125, . . . , and divide the first differences by 5, the second by 5^2, the third by 5^3, . . . (or, which comes to the same thing, multiply them respectively by ·2, by ·04, by ·008, &c.), and these quotients (or products) will be what I call the *mean differences* of the corresponding orders." Briggs then gives examples of the manner of obtaining them; and afterwards proceeds to *correct* them, so as to be able to employ them in his investigations.

11. Having found that the fifth differences of the logarithms of the numbers 2,115, 2,120, 2,125, . . . , are constant, Briggs observes that the *mean* differences of the fourth and fifth orders cannot be *corrected;* for, he says, the sixth and seventh differences are *nil;* adding, "*omnis autem differentiarum correctio fit per subductionem differentiarum alternarum magis remotarum et correctarum*" (every correction of the differences is made by subtracting the alternate corrected differences of higher orders) (Preface, p. 27, near the bottom). This passage appears at first sight to involve a vicious circle, inasmuch as it requires, for the calculation of the corrections, that we should have differences already corrected. But this difficulty disappears when we notice that Briggs begins by obtaining differences which, requiring no correction, may be said to be already corrected; so that when he subsequently makes all these corrections, commencing with the highest order, he obtains successively the corrected differences he requires, whose order in effect *alternates* by increasing at each step by two units (See Articles 15 to 18 and 28 below, and compare them).

Briggs continues :—"Thus the subtraction of the seventh differences corrects the fifth differences, that of the sixth differences corrects the fourth, and so on." Therefore, in the case under consideration, we may take the *mean* fourth and fifth differences for the *corrected* fourth and fifth differences.

As to the mean third differences, we are to correct them by the subtraction of *three* corrected fifth differences, and Briggs gives a numerical example.

The mean second differences are corrected by the subtraction of *two* corrected fourth differences, and also of $\frac{7}{5}$ of the sixth difference, if it have any sensible value. Briggs here gives a fresh example.

Lastly, we are to correct the mean first differences by deducting therefrom *one* corrected third difference, and $\frac{1}{5}$ of the fifth difference; and Briggs gives another calculation as an example.

"Such," adds Briggs, "is the course to be pursued in correcting the differences, whatever may be their order, commencing always with the highest order." He thereupon gives a table showing all the subtractions required to correct the successive mean differences of every order from the first to the twentieth. It shows, for example, that the difference of the fourth order is corrected by the subtraction of the following multiples of the higher corrected dif-

ferences, each difference being represented by the exponent of its order placed within brackets:

$$\begin{bmatrix} 4(6)+6{\cdot}8(8)+6{\cdot}4(10)+3{\cdot}64(12)+1{\cdot}28(14) \\ +{\cdot}272(16)+{\cdot}032(18)+{\cdot}0016(20) \end{bmatrix}.$$

12. Briggs then gives some practical suggestions, intended to prevent confusion in the application of his rules; and, as an example, he finds the logarithms of the eight numbers comprised between 2,115 and 2,120 and between 2,120 and 2,125. For this purpose he forms a table, in which are arranged in different places both the given logarithms and their differences of the first four orders, corrected as previously explained; those of the fifth order being neglected as insignificant. He finally states that the required logarithms will be found by *adding* the fourth differences to the third, the sum of these to the second differences, the new sum to the first differences, and the sum thus found to the preceding logarithm.

But this table appears complicated and badly arranged, if we may venture to say so. For we must *subtract*, and not add the differences, since the differences of the logarithms of increasing numbers decrease in value; and Briggs himself does this. It is, moreover, impossible not to observe the needless fulness of a process by which, the *mean* differences of the fourth order being reduced to two figures in the thirteenth and fourteenth place of decimals, we are told to correct them by means of differences smaller in themselves, which, however, have been previously computed to the sixth, eighth, ... twentieth, place of decimals.

It is not our object here to praise this process, which, as we have just seen, Briggs greatly modifies in his actual computations, but only to explain and prove it. For this is all which the author has given by way of explanation, and that is but little. He only adds that this method, which may be called *quintisection*, applies, with the requisite alterations, to the steps necessary for effecting *trisection* and *septisection*, but that he greatly prefers the method of *quintisection*.

§ III.

13. If Briggs had been in possession of the formula (F), he might have thought of using it to accomplish the object he had in view in his method of *quintisection*. In fact, making $h=5$, and and treating $\Delta^5 u$ as a constant quantity, the quotients $\frac{\Delta^n u_0}{h^n}$, in that

formula would be his *mean differences.* For brevity, write Δ^n for $\frac{\Delta^n u_0}{h^n}$; then applying (F) to find the values of $u_1, u_2, u_3, \ldots$, as far as terms involving Δ^5, we might find $\delta u_0, \delta u_1, \delta u_2, \ldots, \delta^2 u_0, \delta^2 u_1, \ldots$, in terms of $\Delta^1, \Delta^2, \ldots, \Delta^5$, whence, conversely, we could deduce the values of $\Delta^1, \Delta^2, \ldots, \Delta^5$, in terms of $\delta u_0, \delta^2 u_0, \ldots$ But there is no reason to believe that Briggs was acquainted with that formula; and moreover, in following the course just pointed out, he would have encountered long and complicated calculations wholly devoid of symmetry.

14. The formula (D) of art. 4, which is much more simple, was more likely to have suggested itself to him, as offering great advantages for the object he had in view. In fact, when the variable receives successive increments of 5, that formula gives

$$(A)\quad\begin{cases}\Delta u_0 = u_5 - u_0\\ \Delta^2 u_0 = u_{10} - u_5 - (u_5 - u_0) = \mathrm{P},\ \text{suppose}\\ \Delta^3 u_0 = u_{15} - u_{10} - (u_{10} - u_5) - \mathrm{P} = \mathrm{Q} - \mathrm{P}\\ \Delta^4 u_0 = u_{20} - u_{15} - (u_{15} - u_{10}) - 2\mathrm{Q} + \mathrm{P} = \mathrm{R} - \mathrm{Q} - (\mathrm{Q} - \mathrm{P})\\ \Delta^5 u_0 = u_{25} - u_{20} - (u_{20} - u_{15}) - 3\mathrm{R} + 3\mathrm{Q} - \mathrm{P}\\ \qquad = \mathrm{S} - \mathrm{R} - 2(\mathrm{R} - \mathrm{Q}) + (\mathrm{Q} - \mathrm{P})\end{cases}$$

On the other hand, the formula (E) gives, putting $x=5$,

$$u_5 - u_0 = 5(\delta u_0 + 2\delta^2 u_0 + 2\delta^3 u_0 + \delta^4 u_0 + \frac{1}{5}\delta^5 u_0),$$

which may be reduced to

$$(b)\qquad u_5 - u_0 = 5(\delta u_2 + \delta^3 u_1 + \frac{1}{5}\delta^5 u_0).$$

But we have, by the first principles of the method of differences,

$$\begin{aligned} u_5 - u_0 &= \delta u_0 + \delta u_1 + \delta u_2 + \delta u_3 + \delta u_4\\ u_{10} - u_5 &= \delta u_5 + \delta u_6 + \delta u_7 + \delta u_8 + \delta u_9\\ u_{15} - u_{10} &= \delta u_{10} + \delta u_{11} + \delta u_{12} + \delta u_{13} + \delta u_{14}\\ \&c. &= \&c. \end{aligned}$$

Here it is at once seen that each of the preceding equations may be deduced from the preceding one by increasing the variable by 5; and we must therefore get expressions for $u_{10} - u_5$, $u_{15} - u_{10}$, . . . precisely similar to the value given by (*b*) for $u_5 - u_0$. Thus,

$$\begin{aligned} u_{10} - u_5 &= 5(\delta u_7 + \delta^3 u_6 + \frac{1}{5}\delta^5 u_5)\\ u_{15} - u_{10} &= 5(\delta u_{12} + \delta^3 u_{11} + \frac{1}{5}\delta^5 u_{10})\\ \&c. &= \&c. \end{aligned}$$

But we can readily satisfy ourselves that we shall have likewise

$$u_7 - u_2 = 5(\delta u_4 + \delta^3 u_3 + \frac{1}{5}\delta^5 u_2)$$

$$u_{13} - u_8 = 5(\delta u_{10} + \delta^3 u_9 + \frac{1}{5}\delta^5 u_8)$$

$$\&c. = \&c.$$

and we shall be justified in concluding generally that

$$\text{(B)} \qquad u_m - u_{m-5} = 5(\delta u_{m-3} + \delta^3 u_{m-4} + \frac{1}{5}\delta^5 u_{m-5})$$

from which we can easily deduce

$$\text{(C)} \qquad \delta^n(u_m - u_{m-5}) = 5(\delta^{n+1} u_{m-3} + \delta^{n+3} u_{m-4} + \frac{1}{5}\delta^{n+5} u_{m-5}).$$

15. These preliminaries being settled, let us return to the relation (b). It will give, dividing by 5,

$$\frac{u_5 - u_0}{5} = \frac{\Delta u_0}{5} = \Delta^1 = \delta u_2 + \delta^3 u_1 + \frac{1}{5}\delta^5 u_0\,;$$

from which we obtain

$$\delta u_2 = \Delta^1 - \delta^3 u_1 - \frac{1}{5}\delta^5 u_0.$$

But this is the first expression in Briggs's table, which has been verified by Legendre.

(16.) The second of the formulæ (A) and the relation (B) give in turn

$$(x) \qquad \Delta^2 u_0 = 5\left\{\delta(u_7 - u_2) + \delta^3(u_6 - u_1) + \frac{1}{5}\delta^5(u_5 - u_0)\right\}.$$

But by means of the relation (C), we get

$$\delta(u_7 - u_2) = 5\left(\delta^2 u_4 + \delta^4 u_3 + \frac{1}{5}\delta^6 u_2\right)$$

$$\delta^3(u_6 - u_4) = 5\left(\qquad \delta^4 u_3 + \delta^6 u_2 + \frac{1}{5}\delta^8 u_1\right)$$

$$\frac{1}{5}\delta^5(u_5 - u_0) = 5\left(\qquad \frac{1}{5}\delta^6 u_2 + \frac{1}{5}\delta^8 u_1 + \frac{1}{5^2}\delta^{10} u_0\right)$$

Adding and multiplying by 5,

$$\Delta^2 u_0 = 5^2(\delta^2 u_4 + 2\delta^4 u_3 + 1{\cdot}4\delta^6 u_2 + {\cdot}4\delta^8 u_1 + {\cdot}04\delta^{10} u_0);$$

or, dividing by 5^2 and transposing,

$$(2) \qquad \delta^2 u_4 = \Delta^2 - 2\delta^4 u_3 - 1{\cdot}4\delta^6 u_2 - {\cdot}4\delta^3 u_1 - {\cdot}04\delta^{10} u_0.$$

This is the second formula in Briggs's table, which has also been verified by Legendre.

17. The third of the formulæ (A) and the relation (B) will give similarly

$$\Delta^3 u_0 = 5\left\{\delta(u_{12}-u_7)+\delta^3(u_{11}-u_6)+\frac{1}{5}\delta^5(u_{10}-u_5)\right\}-\Delta^2 u_0;$$

or, by virtue of (C) and substituting for $\Delta^2 u_0$ its value as given by (x),

$$(y)\quad \Delta^3 u_0 = 5^2\left\{\delta^2(u_9-u_4)+\delta^4(u_8-u_3)+\frac{1}{5}\delta^6(u_7-u_2)\right\}$$

$$+5^2\left\{\delta^4(u_8-u_3)+\delta^6(u_7-u_2)+\frac{1}{5}\delta^8(u_6-u_1)\right\}$$

$$+5^2\left\{\frac{1}{5}\delta^6(u_7-u_2)+\frac{1}{5}\delta^8(u_6-u_1)+\frac{1}{5^2}\delta^{10}(u_5-u_0)\right\}$$

$$=5^2\left\{\delta^2(u_9-u_4)+2\delta^4(u_8-u_3)+\frac{7}{5}\delta^6(u_7-u_2)\right.$$

$$\left.+\frac{2}{5}\delta^8(u_6-u_1)+\frac{1}{5^2}\delta^{10}(u_5-u_0)\right\}$$

$$=5^3\left[\delta^3 u_6+\delta^5 u_5+\frac{1}{5}\delta^7 u_4+2\left(\delta^5 u_5+\delta^7 u_4+\frac{1}{5}\delta^9 u_3\right)\right.$$

$$+\frac{7}{5}\left(\delta^7 u_4+\delta^9 u_3+\frac{1}{5}\delta^{11} u_2\right)+\frac{2}{5}\left(\delta^9 u_3+\delta^{11} u_2+\frac{1}{5}\delta^{13} u_1\right)$$

$$\left.+\frac{1}{5^2}\left(\delta^{11} u_2+\delta^{13} u_1+\frac{1}{5}\delta^{15} u_0\right)\right]$$

whence, dividing by 5^2, reducing and transposing, we get

$$(3)\quad \delta^3 u_6 = \Delta^3 - 3\delta^5 u_5 - 3{\cdot}6\delta^7 u_4 - 2{\cdot}2\delta^9 u_3 - {\cdot}72\delta^{11} u_2 - {\cdot}12\delta^{13} u_1 - {\cdot}008\delta^{15} u_0,$$

and we thus have the third formula in Briggs's table, the last which has been verified and demonstrated by Legendre.

18. We pass now to the fourth of the formulæ (A). It gives

$$\Delta^4 u_0 = \{u_{20}-u_{15}-(u_{15}-u_{10})\}-\Delta^3 u_0-(\Delta^3 u_0-\Delta^2 u_0);$$

whence, by means of (B) and (C), substituting the values of $\Delta^2 u_0$ and $\Delta^3 u_0$, as found from (x) and (y),

$$\Delta^4 u_0 = 5\left\{\delta(u_{17}-u_{12})+\delta^3(u_{16}-u_{11})+\frac{1}{5}\delta^5(u_{15}-u_{10})\right.$$

$$\left.-\delta(u_{12}-u_7)-\delta^3(u_{11}-u_6)-\frac{1}{5}\delta^5(u_{10}-u_5)\right\}$$

$$-5\left\{\delta(u_{12}-u_7)+\delta^3(u_{11}-u_6)+\frac{1}{5}\delta^5(u_{10}-u_5)\right.$$

$$\left.-\delta(u_7-u_2)-\delta^3(u_6-u_1)-\frac{1}{5}\delta^5(u_5-u_0)\right\}$$

$$=5^2\left\{\delta^2(u_{14}-u_9)+\delta^4(u_{13}-u_8)+\frac{1}{5}\delta^5(u_{12}-u_7)\right.$$
$$\left.-\delta^2(u_9-u_4)-\delta^4(u_8-u_3)-\frac{1}{5}\delta^6(u_7-u_2)\right\}$$
$$+5^2\left\{\delta^4u_{13}-u_8)+\delta^6(u_{12}-u_7)+\frac{1}{5}\delta^8(u_{11}-u_6)\right.$$
$$\left.-\delta^4(u_8-u_3)-\delta^6(u_7-u_2)-\frac{1}{5}\delta^8(u_6-u_1)\right\}$$
$$+5^2\left\{\frac{1}{5}\delta^6(u_{12}-u_7)+\frac{1}{5}\delta^8(u_{11}-u_6)+\frac{1}{5^2}\delta^{10}(u_{10}-u_5)\right.$$
$$\left.-\frac{1}{5}\delta^6(u_7-u_2)-\frac{1}{5}\delta^8(u_6-u_1)-\frac{1}{5^2}\delta^{10}(u_5-u_0)\right\}$$

$$=5^3\left[\delta^3u_{11}+\delta^5u_{10}+\frac{1}{5}\delta^7u_9+\delta^5u_{10}+\delta^7u_9+\frac{1}{5}\delta^9u_8\right.$$
$$+\frac{1}{5}\left(\delta^7u_9+\delta^9u_8+\frac{1}{5}\delta^{11}u_7\right)-\left(\delta^3u_6+\delta^5u_5+\frac{1}{5}\delta^7u_4\right)$$
$$-\left(\delta^5u_5+\delta^7u_4+\frac{1}{5}\delta^9u_3\right)-\frac{1}{5}\left(\delta^7u_4+\delta^9u_3+\frac{1}{5}\delta^{11}u_2\right)$$
$$+\delta^5u_{10}+\delta^7u_9+\frac{1}{5}\delta^9u_8+\delta^7u_9+\delta^9u_8+\frac{1}{5}\delta^{11}u_7$$
$$+\frac{1}{5}\left(\delta^9u_8+\delta^{11}u_7+\frac{1}{5}\delta^{13}u_6\right)-\left(\delta^5u_5+\delta^7u_4+\frac{1}{5}\delta^9u_3\right)$$
$$-\left(\delta^7u_4+\delta^9u_3+\frac{1}{5}\delta^{11}u_2\right)-\frac{1}{5}\left(\delta^9u_3+\delta^{11}u_2+\frac{1}{5}\delta^{13}u_1\right)$$
$$+\frac{1}{5}\left(\delta^7u_9+\delta^9u_8+\frac{1}{5}\delta^{11}u_7\right)+\frac{1}{5}\left(\delta^9u_8+\delta^{11}u_7+\frac{1}{5}\delta^{13}u_6\right)$$
$$+\frac{1}{5^2}\left(\delta^{11}u_7+\delta^{13}u_6+\frac{1}{5}\delta^{15}u_5\right)-\frac{1}{5}\left(\delta^7u_4+\delta^9u_3+\frac{1}{5}\delta^{11}u_2\right)$$
$$\left.-\frac{1}{5}\left(\delta^9u_3+\delta^{11}u_2+\frac{1}{5}\delta^{13}u_1\right)-\frac{1}{5^2}\left(\delta^{11}u_2+\delta^{13}u_1+\frac{1}{5}\delta^{15}u_0\right)\right]$$

$$=5^3\left\{\delta^3(u_{11}-u_6)+3\delta^5(u_{10}-u_5)+\frac{18}{5}\delta^7(u_9-u_4)+\frac{11}{5}\delta^9(u_8-u_3)\right.$$
$$\left.+\frac{18}{5^2}\delta^{11}(u_7-u_2)+\frac{3}{5^2}\delta^{13}(u_6-u_1)+\frac{1}{5^3}\delta^{15}(u_5-u_0)\right\}$$
$$=5^4\left\{\delta^4u_8+4\delta^6u_7+6\cdot8\delta^8u_6+6\cdot4\delta^{10}u_5+3\cdot64\delta^{12}u_4+1\cdot28\delta^{14}u_3\right.$$
$$\left.+\cdot272\delta^{16}u_2+\cdot332\delta^{18}u_1+\cdot0016\delta^{20}u_0\right\}$$

whence, dividing by 5^4 and transposing,

$$(4)\qquad \delta^4u_8=\Delta^4-4\delta^6u_7-6\cdot8\delta^8u_6-6\cdot4\delta^{10}u_5-3\cdot64\delta^{12}u_4$$
$$-1\cdot28\delta^{14}u_3-\cdot272\delta^{16}u_2-\cdot032\delta^{18}u_1-\cdot0016\delta^{20}u_0;$$

and this is in effect the fourth formula in Briggs's table, which we have inserted at art. 11, from page 29 of his Preface.*

19. We shall not proceed any further with these calculations, which present no difficulty. Their law is evident, and they are eminently symmetrical. It must be admitted they are simple, although far from brief. All the operations are purely arithmetical, and depend on the proper use of the relations (B) and (C), applied to two direct consequences of the formulæ (D) and (E). The minute resolutions required return with such symmetry that we can almost write down the results mechanically.

We therefore think it highly probable that this was the course followed by Briggs. It does not presuppose the use of any knowledge superior to that of a contemporary of Kepler and Harriot, whose works appeared about fifteen or twenty years before the more advanced era of Descartes and Fermat.

But, as we have previously stated (arts. 9 and 12), the process appears to us rather curious than useful. There appears to be nothing to guide the computer as to the limit he should prescribe to himself in making his corrections; and, moreover, the differences of various orders having reference to different terms in the original series might give considerable trouble in practice.†

20. On the contrary, nothing can be more lucid and convenient than the method published in 1670 by Mouton; and although, in the last century, two different demonstrations of it have been given, we trust that the more simple and complete exposition we are about to give in the following section may not be considered superfluous.

§ IV.

21. "Given a series of quantities such that the differences of any given order are constant, to find any number whatever of intermediate values that shall follow the same law."

Mouton was the first person who regarded interpolation from this point of view. The process published by him, after Regnauld, is a general one, but very long. He solves the problem by showing how to find by means of the differences of the original quantities, those of the interpolated terms; these terms being then found from the differences by successive additions. Nothing can be simpler than this process.

* M. Maurice's demonstration fails to exhibit the general law of this formula for $\delta^4 u_8$. This defect we hope to supply in our next number.—Ed. *J. I. A.*

† It appears to us that M. Maurice does not do full justice to Briggs's method in consequence of his not observing that the differences $\delta^4 u_8$, $\delta^6 u_7$, $\delta^8 u_6$, $\delta^{10} u_5$ .... are all "central differences" opposite the same value, u_{10}.—Ed. *J. I. A.*

The only difficulty then is to find these differences of the interpolated terms, the number $(h-1)$ of the terms interpolated and the order (n) of the constant differences in the given series, being any whatever.

To solve it, Mouton and Regnauld had arrived at the following principle, probably suggested to them by the examination of several particular cases:—*When the values of a function corresponding to equidistant values of the variable have their differences of any given order constant, then the values interpolated between them at equal intervals will also have their differences of the same order constant.*

This principle is true, but only in the case of rational and integral functions, as in Mouton's problem, and as we will now demonstrate.

22. Let u be a function of x, such that

$$u=\alpha x^m+\beta x^p+\gamma x^q+\ldots.$$

where the indices are all positive integers, and $m>p>q>\ldots$

Let us consider the relation $u=\alpha x^m$. Since m is the highest power of x in the value of u, the consideration of the other terms is useless for our purpose. Supposing $\delta x=h$, we shall have

$$\Delta u=\alpha\{(x+h)^m-x^m\}=\alpha mhx^{m-1}+\mathrm{A}x^{m-2}+\mathrm{B}x^{m-3}+\ldots.$$

from which we deduce

$$\Delta^2u=\alpha mh\Delta x^{m-1}+\mathrm{A}\Delta x^{m-2}+\ldots.$$

and consequently

$$\Delta^2u=\alpha mh(m-1)hx^{m-2}+\mathrm{A}'x^{m-2}+\ldots.$$

In the same manner we shall find

$$\Delta^3u=\alpha mh(m-1)h(m-2)hx^{m-3}+\mathrm{A}''x^{m-4}+\ldots.$$

$$\vdots$$

$$\Delta^nu=\alpha m(m-1)(m-2)\ldots(m-n+1)h^nx^{m-n}+\mathrm{A}^{(n-1)}x^{m-n-1}+\ldots$$

Now, if $m=n$, since $n>p>q>\ldots$, we see that if we had considered the complete value of u, all the terms of Δ^nu after the first would necessarily have disappeared; so that we should have had simply

$$\Delta^nu=n(n-1)(n-2)\ldots.2\alpha h^n$$

for the value of the *constant* difference; for $\Delta^{n+1}u=0$.

It is also evident that if, taking $\delta x=1$, we represent by δu the difference $(x+1)^m-x^m$, we should find in just the same way

$$\delta^nu=n(n-1)(n-2)\ldots.2.1.\alpha,$$

and we should consequently have

$$\delta^n u = \frac{\Delta^n u}{h^n}.$$

Thus the supposed principle is demonstrated, and the value of $\delta^n u$ is found.

23. Assuming then that we have a series of quantities, u_0, u_h, u_{2h}, . . . , which give the differences Δu_0, $\Delta^2 u_0$, $\Delta^3 u_0$, . . . $\Delta^n u_0$, of which the last is constant, we see from what precedes that we should know also $\delta^n u_0$, which relates to the series u_0, u_1, u_2, . . . u_h, . . . and is, like $\Delta^n u_0$, constant. Having also u_0, if we could find δu_0, $\delta^2 u_0$, . . . $\delta^{n-1} u_0$, we might by simple additions obtain the quantities u_1, u_2, u_3, . . . , u_{h-1}, . . .

To establish this, let us confine our attention, like Briggs and Mouton, to the case when $n=h=5$, and bear in mind that in general

$$(m) \qquad u_{m+1} = u_m + \delta u_m.$$

Thus, when u represents a number, it will be seen that each value is formed by the addition of the preceding one and of its difference; and as we know that $u_m = (1+\delta)^m u_0$, it is clear that we shall have

$$\delta u_m = u_{m+1} - u_m = \{(1+\delta)-1\}\{(1+\delta)^m u_0\} = \delta(1+\delta)^m u_0.$$

Consequently, δu_m will be expressed in terms of δu_0, and its successive differences; so that we shall have, m being any whole number,

$$\delta u_m = u_{m+1} - u_m = \delta u_0 + m\delta^2 u_0 + \frac{m(m-1)}{1\,2}\delta^3 u_0 + \frac{m(m-1)(m-2)}{1.2.3}\delta^4 u_0 + \ldots$$

It will therefore be sufficient, as already stated, to know how to express the differences δu_0, $\delta^2 u_0$, $\delta^3 u_0$, $\delta^4 u_0$, in terms of the differences Δ ($\delta^5 u_0$ being a constant and its value known), in order to be able to calculate all the interpolated numbers.

Assume for a moment that we know these four quantities, δu_0, . . . $\delta^4 u_0$, and we shall see how easily we may obtain the interpolated numbers, by virtue of an obvious consequence of the relation (m) which gives

$$(n) \qquad \delta^k u_{m+1} = \delta^k u_m + \delta^{k+1} u_m.$$

24. Let us consider the following table, the law of which is evident :—

$$\begin{array}{llllll}
u_0, & & & & & \\
u_1 & \delta u_0, & & & & \\
u_2 & \delta u_1 & \delta^2 u_0, & & & \\
u_3 & \delta u_2 & \delta^2 u_1 & \delta^3 u_0, & & \\
u_4 & \delta u_3 & \delta^2 u_2 & \delta^3 u_1 & \delta^4 u_0, & \\
u_5 & \delta u_4 & \delta^2 u_3 & \delta^3 u_2 & \delta^4 u_1 & \delta^5 u_0 \text{ (=constant, by hypothesis)} \\
u_6 & \delta u_5 & \delta^2 u_4 & \delta^3 u_3 & \delta^4 u_2 & \delta^5 u_0, \\
u_7 & \delta u_6 & \delta^2 u_5 & \delta^3 u_4 & \delta^4 u_3 & \delta^5 u_0, \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Every term in this table will be known, if we know only those at the end of each horizontal line. For, by virtue of the relation (n), in accordance with which it has been formed, each term being equal to the sum of the two which stand respectively above it and on the right of it, we see that

$$\begin{aligned}
u_1 &= u_0 + \delta u_0 \\
\delta u_1 &= \delta u_0 + \delta^2 u_0 \\
u_2 &= u_1 + \delta u_1 \\
\delta^2 u_1 &= \delta^2 u_0 + \delta^3 u_0 \\
\delta u_2 &= \delta u_1 + \delta^2 u_1 \\
u_3 &= u_2 + \delta u_2 \\
\delta^3 u_1 &= \delta^3 u_0 + \delta^4 u_0 \\
\delta^2 u_2 &= \delta^2 u_1 + \delta^3 u_1 \\
\delta u_3 &= \delta u_2 + \delta^2 u_2 \\
u_4 &= u_3 + \delta u_3
\end{aligned}$$

and so on *ad infinitum*. It is furthermore easy to see how we must proceed if the constant differences are of either a higher or a lower order than $\delta^5 u_0$.

Thus, each of the interpolated values will be formed by adding to the previous one its difference, which itself will be obtained by simple, regular and successive additions, and will involve δu_0, $\delta^2 u_0$, $\delta^3 u_0 \ldots$ up to the order for which these differences are constant.

25. We have lastly to find $\delta u_0, \delta^2 u_0, \ldots \delta^n u_0$, in terms of Δu_0, $\Delta^2 u_0, \ldots, \Delta^n u_0$, supposing that $\Delta^n u_0$ is constant.

Let there be two series of values of the function, in one of which every value is known, but in the other only every hth value is known, namely:—

$$\begin{array}{lllll}
v_0, & v_1, & v_2, & v_3, \ldots\ldots v_x, & \\
u_0, u_1, u_2, \ldots\ldots u_h, & u_{h+1}, \ldots u_{2h}, & u_{2h+1}, \ldots\ldots u_{3h}, & \ldots\ldots u_{xh}, \ldots\ldots &
\end{array}$$

It is clear that if $v_0 = u_0$, $v_1 = u_h$, we shall have generally, $v_x = u_{xh}$.

Denoting by δ the differences of u, and by Δ the differences of v, we have

$$u_1 = u_0 + \delta u_0, \qquad v_1 = v_0 + \Delta v_0 = u_0 + \Delta v_0,$$

whence

$$\Delta v_0 = v_1 - u_0 = u_h - u_0.$$

We therefore have, since $u_h = (1+\delta)^h u_0$,

$$\Delta v_0 = \{(1+\delta)^h - 1\} u_0.$$

Let us confine ourselves, for simplicity, to the case of $h=5$. We shall see however that the course of procedure is quite general. Let us also put $(1+\delta)^5 = \beta$. We shall then easily get

$$\begin{array}{lll}
v_0 = u_0 & v_1 - v_0 = \Delta v_0 = (\beta-1)u_0 = \Delta u_0 & \Delta v_1 - \Delta v_0 = \Delta^2 v_0 = (\beta-1)^2 u_0 = \Delta^2 u_0 \\
v_1 = u_h \ = \beta u_0 & \Delta v_1 = \beta(\beta-1)u_0 & \Delta^2 v_1 = (\beta-1)^3 u_0 \\
v_2 = u_{2h} = \beta^2 u_0 & \Delta v_2 = \beta^2(\beta-1)u_0 & \Delta^2 v_2 = (\beta-1)^4 u_0 \\
v_3 = u_{3h} = \beta^3 u_0 & \Delta v_3 = \beta^3(\beta-1)u_0 & \Delta^2 v_3 = (\beta-2)^5 u_0 \\
v_4 = u_{4h} = \beta^4 u_0 & \Delta v_4 = \beta^4(\beta-1)u_0 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
v_5 = u_{5h} = \beta^5 u_0 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot & &
\end{array}$$

$$\begin{array}{lll}
\Delta^2 v_1 - \Delta^2 v_0 = \Delta^3 v_0 = (\beta-1)^3 u_0 = \Delta^3 u_0 & \Delta^3 v_1 - \Delta^3 v_0 = \Delta^4 v_0 = (\beta-1)^4 u_0 = \Delta^4 u_0 & \Delta^4 v_1 - \Delta^4 v_0 = \Delta^5 v_0 = (\beta-1)^5 u_0 = \Delta^5 u_0 \\
\Delta^3 v_1 = \beta(\beta-1)^3 u_0 & \Delta^4 v_1 = \beta(\beta-1)^4 u_0 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\Delta^3 v_2 = \beta^2(\beta-1)^3 u_0 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & &
\end{array}$$

We have thus shown that

$$\begin{aligned}
\Delta v_0 &= \Delta u_0 = (\beta-1)u_0 = \{1+\delta)^5 - 1\}u_0 \\
\Delta^2 v_0 &= \Delta^2 u_0 = (\beta-1)^2 u_0 = \{1+\delta)^5 - 1\}^2 u_0 \\
\Delta^3 v_0 &= \Delta^3 u_0 = (\beta-1)^3 u_0 = \{1+\delta)^5 - 1\}^3 u_0 \\
\Delta^4 v_0 &= \Delta^4 u_0 = (\beta-1)^4 u_0 = \{1+\delta)^5 - 1\}^4 u_0 \\
\Delta^5 v_0 &= \Delta^5 u_0 = (\beta-1)^5 u_0 = \{1+\delta)^5 - 1\}^5 u_0
\end{aligned}$$

and it is clear that, whatever integers m and h may be, we shall have

$$\Delta^m u_0 = \{(1+\delta)^h - 1\}^m u_0;$$

whence expanding $\{(1+\delta)^h - 1\}$, we have

$$\text{(P)} \qquad \left[h\delta + \frac{h(h-1)}{1.2}\delta^2 + \frac{h(h-1)(h-2)}{1.2.3}\delta^3 + \ldots\right]^m$$
$$= h^m\delta^m + \mathrm{A}\delta^{m+1} + \mathrm{B}\delta^{m+2} + \mathrm{C}\delta^{m+3} + \ldots$$

If then we can by any means determine the numerical values of the coefficients A, B, C, . . . , we shall only have to multiply the second member of (P) by u_0, and we shall have $\Delta^m u_0$ in terms of $\delta^m u_0$, $\delta^{m+1} u_0$, . . . As to the determination of those coefficients, we shall see that the law of their formation is very simple; and we confine ourselves here to remarking that if the differences δ^m be constant, the differences of higher orders will vanish; so that we again get $\Delta^m u_0 = h^m \delta^m u_0$.

26. But the problem is not to find the differences Δ in terms of the differences δ; it is the converse problem we have to solve.

For that purpose, since $v_1=(1+\Delta)v_0=u_h=(1+\delta)^h u_0$, we get

$$(1+\delta)u_0=(1+\Delta)^{\frac{1}{h}}v_0;$$

or, since $u_0=v_0$,

$$\delta u_0=\{(1+\Delta)^{\frac{1}{h}}-1\}u_0.$$

Reasoning as in art. 25, we shall have

$$\delta^m u_0=\{(1+\Delta)^{\frac{1}{h}}-1\}^m u_0.$$

Hence, when $m=1$, we have

$$\delta u_0=\{1+\Delta)^{\frac{1}{h}}-1\}u_0=\left\{\frac{\Delta}{h}+\frac{1(1-h)}{1.2}\cdot\frac{\Delta^2}{h^2}+\frac{1(1-h)(1-2h)}{1.2.3}\cdot\frac{\Delta^3}{h^3}+\dots\right\}u_0.$$

But, if we take $h=5$, suppose $\Delta^5 u_0$ constant, and for brevity put k for $\frac{\Delta}{5}$, we can very easily calculate the numerical values of the coefficients of $k^2 \dots k^5$, and we thus get

(1) $$\delta u_0=(k+ak^2+bk^3+ck^4+dk^5)u_0.$$

But $\delta^2 u_0$ will depend on the square of the value of δ thus found; and since we need not consider higher powers of k than k^5, putting $k+ak^2=\alpha$, $bk^3+ck^4+dk^5=\beta$, we see at once that in the expansion of $\delta^2 u_0=(\alpha+\beta)^2 u_0$, we may confine our attention to the following

$$\delta^2 u_0=(\alpha^2+2\alpha\beta)u_0,$$

which gives

(2) $$\delta^2 u_0=\{k^2+2ak^3+(a^2+2b)k^4+2(ab+c)k^5\}u_0.$$

In the same way

$$\delta^3 u_0=(\alpha+\beta)^3 u_0$$

reduces to

$$\delta^3 u_0=(\alpha^3+3\alpha^2\beta)u_0,$$

whence we obtain

(3) $$\delta^3 u_0=\{k^3+3ak^4+3(a^2+b)k^5\}u_0.$$

In the same way, $\delta^4 u_0=\alpha^4 u_0$ leads to

(4) $$\delta^4 u_0=(k^4+4ak^5)u_0,$$

and lastly

(5) $$\delta^5 u_0=k^5 u_0=\frac{\Delta^5 u_0}{5^5},$$

as we have already more than once seen.

27. The actual calculation of all these coefficients is as simple as it is rapid. Thus, when $h=5$, we have

$$\frac{1(1-h)}{1.2}=-2=a, \qquad a\frac{1-2h}{3}=+6=b,$$

$$b\frac{1-3h}{4}=-21=c, \qquad c\frac{1-4h}{5}=+79{\cdot}8=d;$$

whence

$$2a=-4, \qquad a^2+2b=+16, \qquad 2(ab+c)=-66,$$

$$3a=-6, \qquad 3(a^2+b)=+30, \qquad 4a=-8.$$

Thus, putting for k its value $\frac{\Delta}{5}$, we shall have

$$(1) \qquad \delta u_0=\frac{\Delta u_0}{5}-2\frac{\Delta^2 u_0}{5^2}+6\frac{\Delta^3 u_0}{5^3}-21\frac{\Delta^4 u_0}{5^4}+79{\cdot}8\frac{\Delta^5 u_0}{5^5}$$

$$(2) \qquad \delta^2 u_0=\frac{\Delta^2 u_0}{5^2}-4\frac{\Delta^3 u_0}{5^3}+16\frac{\Delta^4 u_0}{5^4}-66\frac{\Delta^5 u_0}{5^5}$$

$$(3) \qquad \delta^3 u_0=\frac{\Delta^3 u_0}{5^3}-6\frac{\Delta^4 u_0}{5^4}+30\frac{\Delta^5 u_0}{5^5}$$

$$(4) \qquad \delta^4 u_0=\frac{\Delta^4 u_0}{5^4}-8\frac{\Delta^5 u_0}{5^5}$$

$$(5) \qquad \delta^5 u_0=\frac{\Delta^5 u_0}{5^5}$$

Consequently, since we are supposed to know the numerical values of $\Delta u_0, \ldots \Delta^5 u_0$, we shall very easily find the numerical values of $\delta u_0, \delta^2 u_0, \ldots \delta^5 u_0$; and all that is necessary to form the table in art. 24; so that, the origin u_0 being given, all the interpolated terms will be found by very simple successive additions.

We further notice that while it will never be necessary to consider the differences of a higher order than the seventh or eighth, even in these extreme cases, by following the same process, we shall readily be able to obtain all the numbers necessary for the process of interpolation. As to the value of h, however large it may be, no difficulty can arise.

28. It will be interesting to compare the five formulæ above given with those of Briggs, demonstrated in arts. 15 to 18.

But since Briggs has considered $\delta^m u_n$ we must remember that since

$$u_n=(1+\delta)^n u_0,$$

we have also

$$(\delta) \qquad \delta^m u_n=\delta^m(1+\delta)^n u_0.$$

We must also bear in mind that having proved (art. 25) that

the lowest power of k or $\frac{\Delta}{5}$ in the value of $\delta^m u_0$ is m, it will be useless to carry the expansion of the formula (δ) beyond $\delta^5 u_0$; for $\delta^6 u_0$, $\delta^7 u_0$, &c., will involve no lower order of differences than $\Delta^6 u_0$, $\Delta^7 u_0$, &c., respectively; and these are all zero, because $\Delta^5 u_0$ is constant.

Then, taking the formula (δ), the value (1) of art. 15,

$$\delta u_2 = \Delta^1 - \delta^3 u_1 - \frac{1}{5}\delta^5 u_0$$

will become

$$\delta u_0 + 2\delta^2 u_0 + \delta^3 u_0 = \Delta^1 - \delta^3 u_0 - \delta^4 u_0 - \frac{1}{5}\delta^5 u_0,$$

whence

$$(1') \qquad \delta u_0 = \frac{\Delta u_0}{5} - 2\delta^2 u_0 - 2\delta^3 u_0 - \delta^4 u_0 - \frac{1}{5}\delta^5 u_0.$$

In the same way, the value (2) of art. 16, as far as it need be taken, will be

$$\delta^2 u_4 = \Delta^2 - 2\delta^4 u_3$$

or

$$\delta^2 u_0 + 4\delta^3 u_0 + 6\delta^4 u_0 + 4\delta^5 u_0 = \frac{\Delta^2 u_0}{5^2} - 2\delta^4 u_0 - 6\delta^5 u_0,$$

whence

$$(2') \qquad \delta^2 u_0 = \frac{\Delta^2 u_0}{5^2} - 4\delta^3 u_0 - 8\delta^4 u_0 - 10\delta^5 u_0.$$

The value (3) of art. 17 will similarly give

$$\delta^3 u_6 = \Delta^3 - 3\delta^5 u_5$$

or

$$\delta^3 u_0 + 6\delta^4 u_0 + 15\delta^5 u_0 = \frac{\Delta^3 u_0}{5^3} - 3\delta^5 u_0,$$

whence

$$(3') \qquad \delta^3 u_0 = \frac{\Delta^3 u_0}{5^3} - 6\delta^4 u_0 - 18\delta^5 u_0.$$

The value (4) of art. 18 will give in the same way

$$\delta^4 u_0 = \Delta^4,$$

or

$$\delta^4 u_0 + 8\delta^5 u_0 = \frac{\Delta^4 u_0}{5^4},$$

whence

$$(4') \qquad \delta^4 u_0 = \frac{\Delta^4 u_0}{5^4} - 8\delta^5 u_0.$$

Lastly, Briggs having shown (art. 11) that the last difference needs no correction, it is clear that he would take

$$(5'') \qquad \delta^5 u_0 = \Delta^5 = \frac{\Delta^5 u_0}{5^5}.$$

We shall find then, finally, by successive substitution

$$(4'')\qquad \delta^4 u_0 = \frac{\Delta^4 u_0}{5^4} - 8\frac{\Delta^5 u_0}{5^5}$$

$$(3'')\qquad \delta^3 u_0 = \frac{\Delta^3 u_0}{5^3} - 6\frac{\Delta^4 u_0}{5^4} + 30\frac{\Delta^5 u_0}{5^5}$$

$$(2'')\qquad \delta^2 u_0 = \frac{\Delta^2 u_0}{5^2} - 4\frac{\Delta^3 u_0}{5^3} + 16\frac{\Delta^4 u_0}{5^4} - 66\frac{\Delta^5 u_0}{5^5}$$

$$(1'')\qquad \delta u_0 = \frac{\Delta u_0}{5} - 2\frac{\Delta^2 u_0}{5^2} + 6\frac{\Delta^3 u_0}{5^3} - 21\frac{\Delta^4 u_0}{5^4} + 79{\cdot}8\frac{\Delta^5 u_0}{5^5}$$

and we see that these values $(1'')$ . . . $(5'')$ coincide exactly with the values (1) . . . (5) of art. 27; so that the methods of Briggs and Mouton, notwithstanding their great dissimilarity, lead to identically the same results.

§ V.

29. Hitherto we have supposed the given values to be equidistant; but this is not always the case.

Thus, for example, in observing certain phenomena it sometimes happens that the intervals between the observations are not equal. In that case, the time being taken for the variable, and the results of observation giving the values of the function u_x, those values will correspond to values of the variable, which do not proceed by a constant difference; and the problem consists in finding a general expression for u_x in terms of the given values and one value of x, corresponding to any time we choose.

Let the known values of u_x, corresponding to the values of x, $0, p, q, r, s \ldots$, be $u_0, u_p, u_q, u_r, u_s \ldots$; then we have to find a general expression for u_x, whatever be the value of x.

Reasoning as before, we may assume

$$u_x = u_0 + Bx + Cx(x-p) + Dx(x-p)(x-q) + Ex(x-p)(x-q)(x-r) + \ldots$$

and then denoting by Δ, Δ', Δ'', Δ''' . . . the differences $u_p - u_0$, $u_q - u_0$, $u_r - u_0$, . . . we have

$$u_p - u_0 = \Delta = Bp, \quad u_q - u_0 = \Delta' = Bq + Cq(q-p),$$
$$u_r - u_0 = \Delta'' = Br + Cr(r-p) + Dr(r-p)(r-q) \ldots .$$

Whence we shall find

$$\mathrm{B}=\frac{\Delta}{p}$$

$$\mathrm{C}=\frac{\Delta'}{q(q-p)}+\frac{\Delta}{p(p-q)}$$

$$\mathrm{D}=\frac{\Delta''}{r(r-p)(r-q)}+\frac{\Delta'}{q(q-p)(q-r)}+\frac{\Delta}{p(p-q)(p-r)}$$

$$\mathrm{E}=\frac{\Delta'''}{s(s-p)(s-q)(s-r)}+\frac{\Delta''}{r(r-p)(r-q)(r-s)}+\frac{\Delta'}{q(q-p)(q-r)(q-s)}$$
$$+\frac{\Delta}{p(p-q)(p-r)(p-s)}$$

&c.= &c.

the law of these successive values being simple and obvious. This symmetry, as well as the recurrence of the positive sign which unites all the terms, results from the fact that in the several factors of each denominator we have throughout written that letter first, which is itself one of the factors. Thus, for example, we have written $p(q-p)$ under the form $-p(p-q)$: and similarly with the others.

Having thus obtained the values of the coefficients B, C, D . . . we shall have for the required expression, collecting the terms involving Δ, Δ', Δ'' . . . respectively,

$$(a)\left\{\begin{aligned} u_x=u_0&+\left\{\frac{x}{p}+\frac{x(x-p)}{p(p-q)}+\frac{x(x-p)(x-q)}{p(p-q)(p-r)}+\frac{x(x-p)(x-q)(x-r)}{p(p-q)(p-r)(p-s)}+\ldots\right\}\Delta\\ &+\left\{\frac{x(x-p)}{q(q-p)}+\frac{x(x-p)(x-q)}{q(q-p)(q-r)}+\frac{x(x-p)(x-q)(x-r)}{q(q-p)(q-r)(q-s)}+\ldots\right\}\Delta'\\ &+\left\{\frac{x(x-p)(x-q)}{r(r-p)(r-q)}+\frac{x(x-p)(x-q)(x-r)}{r(r-p)(r-q)(r-s)}+\ldots\right\}\Delta''\\ &+\left\{\frac{x(x-p)(x-q)(x-r)}{s(s-p)(s-q)(s-r)}+\ldots\right\}\Delta'''+\ldots\ldots \end{aligned}\right.$$

30. Let now $\mathrm{X}=x(x-p)(x-q)(x-r)(x-s)\ldots$, and denote $\frac{\mathrm{X}}{x-w}$ by X_w. Then we should have similarly

$$\mathrm{P}_p=p(p-q)(p-r)(p-s)\ldots.$$
$$\mathrm{Q}_q=q(q-p)(q-r)(q-s)\ldots.$$
$$\mathrm{R}_r=r(r-p)(r-q)(r-s)\ldots.$$
$$\mathrm{S}_s=s(s-p)(s-q)(s-r)\ldots.$$

&c.= &c.

If we next consider the coefficients of Δ, Δ', . . . , in the above formula (a) for u_x, we shall notice that the several terms in them can be grouped according to a regular law.

Thus, the first being, for example,

$$\frac{x(x-p)\ldots.(x-t)}{v(v-p)\ldots.(v-t)} \text{ or V},$$

the sum of the first and second will be

$$\mathrm{V}\left(1+\frac{x-v}{v-y}\right)=\mathrm{V}\frac{x-y}{v-y}=\mathrm{V}_1 \text{ suppose.}$$

The sum of the first, second, and third will be

$$\mathrm{V}_1\left(1+\frac{x-v}{v-z}\right)=\mathrm{V}_1\frac{x-z}{v-z}=\mathrm{V}_2 \text{ suppose,}$$

and so on, where we see the simple factor $x-v$ has disappeared entirely. Treating each of the coefficients of Δ, Δ', Δ'', . . . in this way, we can easily see that the value of u_x can be accurately written in the concise form

$$(a') \qquad u_x=u_0+\frac{\mathrm{X}_p}{\mathrm{P}_p}\Delta+\frac{\mathrm{X}_q}{\mathrm{Q}_q}\Delta'+\frac{\mathrm{X}_r}{\mathrm{R}_r}\Delta''+\frac{\mathrm{X}_s}{\mathrm{S}_s}\Delta'''+\ldots.$$

In this formula, each of the coefficients of Δ, Δ', Δ'', . . . (which we will denote by α, β, γ, δ, . . . for brevity) is easily calculated by logarithms; and the number of the coefficients will be the same as that of the values p, q, r, . . .

31. This expression for u_x has been readily obtained, because we have considered only the differences u_p-u_0, u_q-u_0, u_r-u_0, &c.; but since these differences of necessity increase, the expression is not convergent, and we shall find a more convenient form by introducing instead of Δ, Δ', Δ'' . . . the differences Δu_0, Δu_p, Δu_q, . . . between the successive values of u_x, which will not necessarily increase.

For this purpose, we notice that

$$\begin{aligned}
\Delta\ &=u_p-u_0=\Delta u_0\\
\Delta'&=u_q-u_0=u_q-u_p+u_p-u_0=\Delta u_p+\Delta u_0\\
\Delta''&=u_r-v_0=u_r-u_q+u_q-u_p+u_p-u_0=\Delta u_q+\Delta u_p+\Delta u_0
\end{aligned}$$

and so on.

If, then, we make these substitutions in the above value of u_x, denoting as before the coefficients by α, β, γ, . . . , the formula (a') will take the preferable form

$$\begin{aligned}
(a'') \qquad u_x=u_0&+(\alpha+\beta+\gamma+\delta+\ldots.)\Delta u_0+(\beta+\gamma+\delta+\ldots.)\Delta u_p\\
&+(\gamma+\delta+\ldots.)\Delta u_q+(\delta+\ldots.)\Delta u_r+\ldots.
\end{aligned}$$

32. But we can very easily obtain a symmetrical and more convergent formula. In fact, by the last article

$$(1)\quad \begin{cases} \Delta \;\; = \Delta u_0 \\ \Delta' \;= \Delta u_p + \Delta u_0 \\ \Delta'' = \Delta u_q + \Delta' \\ \Delta''' = \Delta u_r + \Delta'' \\ \Delta^{\text{IV}} = \Delta u_s + \Delta''' \\ \&\text{c.} = \quad \&\text{c.} \end{cases}$$

whence we obtain

$$(2)\quad \begin{cases} \Delta u_0 = \Delta, & \\ \Delta u_p = \Delta' - \Delta, & \Delta u_p - \Delta u_0 = \Delta' - 2\Delta = \Delta^2 u_0, \\ \Delta u_q = \Delta'' - \Delta', & \Delta u_q - \Delta u_p = \Delta'' - 2\Delta' + \Delta = \Delta^2 u_p, \\ \Delta u_r = \Delta''' - \Delta'', & \Delta u_r - \Delta u_q = \Delta''' - 2\Delta'' + \Delta' = \Delta^2 u_q, \\ \Delta u_s = \Delta^{\text{IV}} - \Delta''', & \Delta u_s - \Delta u_r = \Delta^{\text{IV}} - 2\Delta''' + \Delta'' = \Delta^2 u_r, \\ \cdots\cdots & \cdots\cdots\cdots\cdots \end{cases}$$

$$(3)\quad \begin{cases} \Delta^2 u_p - \Delta^2 u_0 = \Delta'' - 3\Delta' + 3\Delta = \Delta^3 u_0, \\ \Delta^2 u_q - \Delta^2 u_p = \Delta''' - 3\Delta'' + 3\Delta' - \Delta = \Delta^3 u_p, \\ \Delta^2 u_r - \Delta^2 u_q = \Delta^{\text{IV}} - 3\Delta''' + 3\Delta'' - \Delta' = \Delta^3 u_q, \\ \cdots\cdots\cdots\cdots \end{cases}$$

$$(4)\quad \begin{cases} \Delta^3 u_p - \Delta^3 u_0 = \Delta''' - 4\Delta'' + 6\Delta' - 4\Delta = \Delta^4 u_0, \\ \Delta^3 u_q - \Delta^3 u_p = \Delta^{\text{IV}} - 4\Delta''' + 6\Delta'' - 4\Delta = \Delta^4 u_p, \\ \cdots\cdots\cdots\cdots \end{cases}$$

$$(5)\quad \begin{cases} \Delta^4 u_p - \Delta^4 u_0 = \Delta^{\text{IV}} - 5\Delta''' + 10\Delta'' - 10\Delta' + 5\Delta = \Delta^5 u_0, \\ \cdots\cdots\cdots\cdots \end{cases}$$

From these we readily conclude that

$$\begin{aligned} \Delta \;\; &= \Delta u_0, \\ \Delta' \; &= \Delta^2 u_0 + 2\Delta u_0, \\ \Delta'' &= \Delta^3 u_0 + 3\Delta^2 u_0 + 3\Delta u_0, \\ \Delta''' &= \Delta^4 u_0 + 4\Delta^3 u_0 + 6\Delta^2 u_0 + 4\Delta u_0, \\ \Delta^{\text{IV}} &= \Delta^5 u_0 + 5\Delta^4 u_0 + 10\Delta^3 u_0 + 10\Delta^2 u_0 + 5\Delta u_0, \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

So that, substituting these values in (a'), it will become

$$\begin{aligned} (a''') \qquad u_x = u_0 &+ [\alpha + 2\beta + 3\gamma + 4\delta + 5\epsilon + \ldots.].\Delta u_0 \\ &+ [\beta + 3\gamma + 6\delta + 10\epsilon + \ldots.].\Delta^2 u_0 \\ &+ [\gamma + 4\delta + 10\epsilon + \ldots].\Delta^3 u_0 + [\delta + 5\epsilon + \ldots].\Delta^4 u_0 \\ &+ [\epsilon + \ldots.].\Delta^5 u_0 + \ldots., \end{aligned}$$

a perfectly regular series, of which the law is evident, the differences of u_0 in general converging rapidly, while the computation demands only, so to speak, that of the numbers α, β, γ, δ, ϵ, . . . which is very easy.

As to the values of $\Delta u_0, \Delta^2 u_0, \ldots \Delta^5 u_0, \ldots$ the equations (1), (2), . . . (5) give each of them in terms of the immediate differences $\Delta, \Delta', \ldots \Delta^{\text{iv}} \ldots$

33. This was not the form in which the first solution of our problem was given by Newton in Lemma V. of the third book of his *Principia.* Laplace, in his *Mécanique Céleste,* has given a solution virtually the same as Newton's, but in a rather different form, which we will now demonstrate.

We will, in the first instance, establish an *Algebraical Lemma,* which for the sake of symmetry we shall find it convenient to use in certain subtractions, and which may prove useful on other occasions.

"Given any number, m, of binomial factors, as $x-a$ or α, $x-b$ or β, $x-c$ or γ, &c. . . . ; and the same number involving y instead of x, but with the same second terms, viz., $y-a$ or A, $y-b$ or B, $y-c$ or C . . . , the difference $\alpha\beta\gamma \ldots - \text{ABC} \ldots$ will be divisible by $x-y$; and the quotient will have two equivalent symmetrical forms."

1°. We see at once that the difference will be a polynomial of the form

$$x^m - y^m - \text{P}(x^{m-1} - y^{m-1}) + \text{Q}(x^{m-2} - y^{m-2}) - \text{R}(x^{m-3} - y^{m-3}) \ldots \pm \text{W}.(x-y),$$

and that this polynomial is divisible by $x-y$;

2°. If we suppose $x-y=z$, it is easy to see that the differences $\alpha-\text{A}, \beta-\text{B}, \gamma-\text{C}, \ldots$ are each equal to z, and that we shall thus have, *ad libitum,*

$$\begin{array}{lllll} \text{either} & (1) & \alpha=\text{A}+z, & \beta=\text{B}+z, & \gamma=\text{C}+z \ldots, \\ \text{or} & (2) & \text{A}=\alpha-z, & \text{B}=\beta-z, & \text{C}=\gamma-z \ldots, \end{array}$$

Thus, according as we employ the relations (1) or (2), and take m equal to 2, 3, 4, . . . , we shall have the following as the differences of the products under consideration:—

Successive values of	According to the relations (1).		According to the relations (2).
$\alpha\beta - \text{AB}$	$= (\text{B}+\alpha)z$	$= \pi z$	$= (\beta + \text{A})z,$
$\alpha\beta\gamma - \text{ABC}$	$= (\text{C}\pi + \alpha\beta)z$	$= \pi_1 z$	$= (\gamma\pi + \text{AB})z,$
$\alpha\beta\gamma\delta - \text{ABCD}$	$= (\text{D}\pi_1 + \alpha\beta\gamma)z$	$= \pi_2 z$	$= (\delta\pi_1 + \text{ABC})z,$
$\alpha\beta\gamma\delta\epsilon - \text{ABCDE}$	$= (\text{E}\pi_2 + \alpha\beta\gamma\delta)z$	$= \pi_3 z$	$= (\epsilon\pi_2 + \text{ABCD})z,$
. . . . .	. . . . .	. . . .	. . . . .

We have thus obtained two sets of values for the quotients π, $\pi_1, \pi_2, \pi_3, \ldots$ the expansion and comparison of which will give the following symmetrical relations:—

$$
\begin{aligned}
\mathrm{B}\ +\alpha\quad &= \pi\ = \beta\ + \mathrm{A},\\
\mathrm{BC} + \mathrm{C}\alpha + \alpha\beta\quad &= \pi_1 = \beta\gamma + \gamma\mathrm{A} + \mathrm{AB},\\
\mathrm{BCD} + \mathrm{CD}\alpha + \mathrm{D}\alpha\beta + \alpha\beta\gamma &= \pi_2 = \beta\gamma\delta + \gamma\delta\mathrm{A} + \delta\mathrm{AB} + \mathrm{ABC},\\
\mathrm{BCDE} + \mathrm{CDE}\alpha + \mathrm{DE}\alpha\beta + \mathrm{E}\alpha\beta\gamma + \alpha\beta\gamma\delta &= \pi_3 = \beta\gamma\delta\epsilon + \gamma\delta\epsilon\mathrm{A} + \delta\epsilon\mathrm{AB}\\
&\qquad + \epsilon\mathrm{ABC} + \mathrm{ABCD},\\
&\cdots\cdots\cdots
\end{aligned}
$$

and it may be observed that either of these two series may be derived from the other, if in that which we assume as given, we write $\alpha, \beta, \gamma, \ldots$ for A, B, C, ..., respectively, and A, B, C, ... for $\alpha, \beta, \gamma, \ldots$, respectively.

34. So much being premised, let $n, p, q, r, s, \ldots$ be values of x to which correspond the following given values of u_x, viz., $u, u', u'', u''', u^{\text{iv}} \ldots$ It is clear that we may assume

$$
\begin{aligned}
u_x = u + \mathrm{B}(x-n) + \mathrm{C}(x-n)(x-p) + \mathrm{D}(x-n)(x-p)(x-q)\\
+ \mathrm{E}(x-n)(x-p)(x-q)(x-r) + \ldots,
\end{aligned}
$$

where the coefficients B, C, D, E, ... are to be determined by means of the given values of u_x.

If we make $n=0$ in the above value of u_x, it becomes identical with the one in art. 29; but we have preferred the above form, because it leads to more symmetrical results.

Now putting x equal to $n, p, q \ldots$ successively, we have

$$
\begin{aligned}
u\ &= u,\\
u'\ &= u + \mathrm{B}(p-n),\\
u''\ &= u + \mathrm{B}(q-n) + \mathrm{C}(q-n)(q-p),\\
u''' &= u + \mathrm{B}(r-n) + \mathrm{C}(r-n)(r-p) + \mathrm{D}(r-n)(r-p)(r-q),\\
u^{\text{iv}} &= u + \mathrm{B}(s-n) + \mathrm{C}(s-n)(s-p) + \mathrm{D}(s-n)(s-p)(s-q)\\
&\qquad + \mathrm{E}(s-n)(s-p)(s-q)(s-r),\\
&\cdots\cdots\cdots
\end{aligned}
$$

From these equations we can deduce by means of the lemma in art. 33,

$$
\begin{aligned}
u' - u\ &= \mathrm{B}(p-n),\\
u'' - u'\ &= \mathrm{B}(q-p) + \mathrm{C}(q-n)(q-p),\\
u''' - u'' &= \mathrm{B}(r-q) + \mathrm{C}(r-q).[(q-n) + (r-p)] + \mathrm{D}(r-n)(r-p)(r-q),\\
u^{\text{iv}} - u''' &= \mathrm{B}(s-r) + \mathrm{C}(s-r).[(r-p) + (s-n)]\\
&\qquad + \mathrm{D}(s-r)[(r-n)(r-p) + (r-n)(s-q) + (s-p)(s-q)]\\
&\qquad + \mathrm{E}(s-n)(s-p)(s-q)(s-r).\\
&\cdots\cdots\cdots
\end{aligned}
$$

Now dividing and using the symbol δ for brevity, we obtain

$$
\begin{aligned}
\frac{u'-u}{p-n} &= \delta u = \mathrm{B},\\
\frac{u''-u'}{q-p} &= \delta u' = \mathrm{B} + \mathrm{C}(q-n),
\end{aligned}
$$

$$\frac{u'''-u''}{r-q}=\delta u''=\mathrm{B}+\mathrm{C}(q-n)+\mathrm{C}(r-p)+\mathrm{D}(r-n)(r-p),$$

$$\frac{u^{\mathrm{IV}}-u'''}{s-r}=\delta u'''=\mathrm{B}+\mathrm{C}(r-p)+\mathrm{C}(s-n)+\mathrm{D}(r-n)(r-p)+\mathrm{D}(r-n)(s-q)$$
$$+\mathrm{D}(s-p)(s-q)+\mathrm{E}(s-n)(s-p)(s-q).$$

. . . .

A similar operation will give also

$$\frac{\delta u'-\delta u}{q-n}=\delta^2 u=\mathrm{C},$$

$$\frac{\delta u''-\delta u'}{r-p}=\delta^2 u'=\mathrm{C}+\mathrm{D}(r-n),$$

$$\frac{\delta u'''-\delta u''}{s-q}=\delta^2 u''=\mathrm{C}+\mathrm{D}(r-n)+\mathrm{D}(s-p)+\mathrm{E}(s-n)(s-p),$$

. . . .

and in the same way we shall find

$$\frac{\delta^2 u'-\delta^2 u}{r-n}=\delta^3 u=\mathrm{D},$$

$$\frac{\delta^2 u''-\delta^2 u'}{s-p}=\delta^3 u'=\mathrm{D}+\mathrm{E}(s-n),$$

. . . .

$$\frac{\delta^3 u'-\delta^3 u}{s-n}=\delta^4 u=\mathrm{E}.$$

. . . .

The other coefficients, if required, would be found in exactly the same way.

Substituting now the values of the four first coefficients, as just found, we have for the value of u_x

$$u_x=u+(x-n).\delta u+(x-n)(x-p).\delta^2 u+(x-n)(x-p)(x-q).\delta^3 u$$
$$+(x-n)(x-p)(x-q)(x-r).\delta^4 u+\&c.;$$

The process of computation by means of this formula is very easy, the numerical values of the quantities δu, $\delta^2 u$, . . . being known, and being in most instances sufficiently small to allow of our confining ourselves to a few of the first terms of the series; a point easily decided in any case. This is the form adopted by Laplace, *Mécanique Céleste,* tom. i., p. 200.

35. Notwithstanding the dissimilarity between this solution and ours, it may be shown that they agree exactly; and we will conclude this paper by proving that such is the case.

If we substitute for δu, $\delta^2 u$, $\delta^3 u$, . . . their values deduced from the mode of their formation, as explained above, we shall find

$$\delta u = \frac{u'}{p-n} + \frac{u}{n-p},$$

$$\delta^2 u = \frac{u''}{(q-n)(q-p)} + \frac{u'}{(p-n)(p-q)} + \frac{u}{(n-p)(n-q)},$$

$$\delta^3 u = \frac{u'''}{(r-n)(r-p)(r-q)} + \frac{u''}{(q-n)(q-p)(q-r)} + \frac{u'}{(p-n)(p-q)(p-r)} + \frac{u}{(n-p)(n-q)(n-r)},$$

. . . .

Nothing can be more symmetrical than these expressions.

Now in order to compare this solution with ours, we must, as already observed, make $n=0$; and we shall then have, for example,

$$\delta^3 u = \mathrm{D} = \frac{u'''}{r(r-p)(r-q)} + \frac{u''}{q(q-p)(q-r)} + \frac{u'}{p(p-q)(p-r)} - \frac{u}{pqr},$$

which, for brevity, we will write

$$\mathrm{D} = \frac{u'''}{\mathrm{R}} + \frac{u''}{\mathrm{Q}} + \frac{u'}{\mathrm{P}} - \frac{u}{pqr}.$$

But in art. 29 we found the following value for D, which ought to be equal to the preceding, since we have put $n=0$,

$$\mathrm{D} = \frac{\Delta''}{\mathrm{R}} + \frac{\Delta'}{\mathrm{Q}} + \frac{\Delta}{\mathrm{P}},$$

or

$$\mathrm{D} = \frac{u'''-u}{\mathrm{R}} + \frac{u''-u}{\mathrm{Q}} + \frac{u'-u}{\mathrm{P}}.$$

We must therefore have

$$\frac{1}{\mathrm{R}} + \frac{1}{\mathrm{Q}} + \frac{1}{\mathrm{P}} = \frac{1}{pqr},$$

which is seen to be the case, when for R, Q, P, we substitute their values.

APPENDIX.

On a case that may occur in astronomical interpolations.

I. When a series of astronomical observations is made, the element observed may be considered as a function of the time, since it varies therewith. Let t denote the time. We may then, as in the calculus of finite differences, represent the value of the element under consideration, whether observed or computed, by u_t; the time t being considered as the independent variable.

Suppose that we know n values of the element, or of the function u_t, viz., a, a', a'', . . . corresponding to the times θ, θ', θ'', . . . which are supposed to be successive and equidistant, and that we wish to deduce a general expression for u_t that shall be true for any value of t not too remote from the period under consideration. The required solution will be possible, and indeed very simple, if the successive differences of the given

values become either constant, or small enough to admit of our neglecting all beyond a certain order.

Suppose, for example, that this is the case with the fourth differences, so that we may neglect the differences of the fifth and higher orders: it will be sufficient to consider five values of u_t, or to have $n=5$.

Also let

$$\theta'-\theta=\theta''-\theta'=\theta'''-\theta''=\theta^{\text{iv}}-\theta'''=\alpha.$$

The common difference of the increasing arithmetical progression $\theta, \theta', \ldots \theta^{\text{iv}}$, will then be α, and it is clear that we may assume

$$u_t=\mathrm{A}+\mathrm{B}(t-\theta)+\mathrm{C}(t-\theta)(t-\theta')+\mathrm{D}(t-\theta)(t-\theta')(t-\theta'')$$
$$+\mathrm{E}(t-\theta)(t-\theta')(t-\theta'')(t-\theta''').$$

The given values of u_t will then lead to the following equations:—

$$\begin{aligned} a &= \mathrm{A}, \\ a' &= a+\mathrm{B}.\alpha, \\ a'' &= a+\mathrm{B}.2\alpha+\mathrm{C}.2\alpha^2, \\ a''' &= a+\mathrm{B}.4\alpha+\mathrm{C}.6\alpha^2+\mathrm{D}.6\alpha^3, \\ a^{\text{iv}} &= a+\mathrm{B}.4\alpha+\mathrm{C}.12\alpha^2+\mathrm{D}.24\alpha^3+\mathrm{E}.24\alpha^4, \end{aligned}$$

and we may thence conclude almost directly, from the known values of the *differences* of any order in terms of given *numbers*, that

$$\mathrm{A}=a;\ \mathrm{B}=\frac{1}{\alpha}\cdot\delta a;\ \mathrm{C}=\frac{1}{1.2.\alpha^2}\cdot\delta^2 a;\ \mathrm{D}=\frac{1}{1.2.3.\alpha^3}\cdot\delta^3 a;\ \mathrm{E}=\frac{1}{1.2.3.4.\alpha^4}\delta^4 a.$$

We shall thus have for the solution sought—

$$(h)\quad u_t=a+\frac{t-\theta}{\alpha}\cdot\delta a+\frac{(t-\theta)(t-\theta')}{1.2\ \alpha^2}\cdot\delta^2 a+\frac{(t-\theta)(t-\theta')(t-\theta'')}{1.2.3.\alpha^3}\cdot\delta^3 a$$
$$+\frac{(t-\theta)(t-\theta')(t-\theta'')(t-\theta''')}{1.2.3.4.\alpha^4}\cdot\delta^4 a.$$

It is also easy from the symmetry to perceive the form this formula would take if n were larger, and we were called upon to consider differences of a higher order than the fourth; and we see readily what terms should then be added to the preceding.

II. Now if we desire to reckon the time t from the middle of the period under consideration, so that $\theta''=0$, and to take the time between two consecutive observations as the unit of time, we must, in the formula (h), put

$$\alpha=1,\quad \theta=-2,\quad \theta'=-1,\quad \theta''=0,\quad \theta'''=+1.$$

That formula will then give

$$(h')\qquad u_t=a+\frac{t+2}{1}\cdot\delta a+\frac{(t+2)(t+1)}{1.2}\cdot\delta^2 a+\frac{(t+2)(t+1)t}{1.2.3}\cdot\delta^3 a$$
$$+\frac{(t+2)(t+1)t(t-1)}{1.2.3.4}\cdot\delta^4 a,$$

where, as before, we have only to do with a and its differences, as found from the five given values of u_t.

III. Stirling (followed by Lacroix, tome iii., pp. 27–8) has given the formula for a similar case under a different form, which he has only been able to generalize by induction. M. Bessel, who has employed the formula

given by Stirling, appears to have demonstrated it in the following manner for the case where $n=5$.

Let $$u_t = A + Bt + Ct^2 + Dt^3 + Et^4.$$

Then making t successively equal to -2, -1, 0, $+1$ and $+2$, we shall have five equations which we may denote respectively by (-2), (-1), (0), $(+1)$ and $(+2)$.

Then
$$\begin{aligned}
(+1)+(-1) &\text{ gives } \frac{a''' + a'}{2} - a'' = C + E;\\
(+2)+(-2) &\quad ,, \quad \frac{a^{\text{iv}} + a}{2} - a'' = 4C + 16E;\\
(0) &\quad ,, \quad a'' = A;\\
(+1)-(-1) &\quad ,, \quad \frac{a''' - a'}{2} = B + D;\\
(+2)-(-2) &\quad ,, \quad \frac{a^{\text{iv}} - a}{2} = 2B + 8D.
\end{aligned}$$

We hence obtain, by means of the well known values of the various orders of differences in terms of the given numbers a, a', . . . a^{iv}, the following values:—

$$A = a'', \quad B = \frac{\delta a' + \delta a''}{2} - \frac{\delta^3 a + \delta^3 a'}{2.6}, \quad C = \frac{\delta^2 a'}{2} - \frac{\delta^4 a}{24},$$
$$D = \frac{\delta^3 a + \delta^3 a'}{2.6}, \quad E = \frac{\delta^4 a}{24}.$$

Substituting these values, and rearranging the terms with reference to the powers of t, we get

$$(g) \quad u_t = a'' + \left[\frac{\delta a' + \delta a''}{2}\right] \cdot \frac{t}{1} + (\delta^2 a') \cdot \frac{t^2}{1.2} + \left[\frac{\delta^3 a + \delta^3 a'}{2}\right] \cdot \frac{t(t^2-1)}{1.2.3}$$
$$+ (\delta^4 a) \cdot \frac{t^2(t^2-1)}{1.2.3.4}.$$

But this formula is subject to the inconvenience of involving differences of several of the given values; while the formula (h') contains only the several differences of the first value a. It also appears much less symmetrical, and less easy to remember: and, above all, it is more difficult of rigorous generalization than the formulæ (h) and (h').

Notwithstanding the apparent difference of the formulæ (h') and (g), it is easy to prove that they only give the same result under different forms.

Another case of astronomical interpolation.

IV. In astronomy the third (if not the second) differences are generally constant, or, at all events, small enough to permit of our neglecting the fourth differences; and the formula (h) then reduces to

$$u_t = a + \frac{t-\theta}{\alpha} \cdot \delta a + \frac{(t-\theta)(t-\theta')}{1.2.\alpha^3} \cdot \delta^2 a + \frac{(t-\theta)(t-\theta')(t-\theta'')}{1.2.3.\alpha^3} \cdot \delta^3 a,$$

in which θ, θ', θ'', θ''', being equidistant values of t, corresponding to the known values a, a', a'', a''', of u_t, we have

$$\alpha = \theta' - \theta = \theta'' - \theta' = \theta''' - \theta''.$$

Now, if the third differences are constant, we take from the tables, or we know from the observations, only the four values a, a', a'', a''', of which

the former two will correspond to times that precede, and the others to times that follow the epoch t for which we are computing, and which must consequently fall between θ' and θ''. It will therefore now be convenient to make $\theta'=0$, whence

$$\theta=-\alpha, \quad \text{and} \quad \theta''=+\alpha.$$

The formula then gives

$$(h'') \quad u_t=a+\frac{t+\alpha}{\alpha}\cdot\delta a+\frac{(t+\alpha)t}{1.2.\alpha^2}\cdot\delta^2 a+\frac{t(t^2-\alpha^2)}{1.2.3.\alpha^3}\cdot\delta^3 a;$$

or,

$$(l)\left\{\begin{aligned} u_t&=a+\delta a+\frac{t}{\alpha}\left[\delta a+\delta^2 a\right]+\frac{t^2-\alpha t}{1.2.\alpha^2}\cdot\delta^2 a+\frac{t(t^2-\alpha^2)}{1.2.3\alpha^3}\cdot\delta^3 a\\ &=a'+\frac{t}{\alpha}\cdot\delta a'+\frac{t(t-\alpha)}{\alpha^2}\cdot\frac{1}{4}\left[\delta^2 a'+\delta^2 a\right]+\frac{t(t-\alpha)}{1.2.\alpha^2}\left[\frac{t+\alpha}{3\alpha}-\frac{1}{2}\right]\cdot\delta^3 a\\ &=a'+\frac{t}{\alpha}\cdot\delta a'+\frac{t(t-\alpha)}{\alpha^2}\cdot\frac{1}{4}\left[\delta^2 a'+\delta^2 a\right]+\frac{t(t-\alpha)(t-\frac{1}{2}\alpha)}{1.2.3.\alpha^3}\cdot\delta^3 a.\end{aligned}\right.$$

This formula (l) coincides exactly with formula (2) of p. 99 of *L'Astronomie pratique* of *M. Francœur*, when we make

$$\alpha=12^h, \text{ whence } \frac{1}{2}\alpha=6^h,$$

and put $\Delta'=\delta a'$, and $\phi=\frac{1}{4}(\delta^2 a'+\delta^2 a)$.

It may also be made to coincide with the formula (3) at the top of p. 100 of the same work. In fact, the correction x or u_t-a', to be applied to the value a', becomes

$$(n) \quad x=\frac{t}{\alpha}\left[\delta a'-\phi+\frac{1}{12}\cdot\delta^3 a\right]+\frac{t^2}{\alpha^2}\left[\phi-\frac{1}{4}\cdot\delta^3 a\right]+\frac{t^3}{\alpha^3}\cdot\frac{1}{6}\cdot\delta^3 a,$$

which is of the following form—

$$x=\mathrm{A}\cdot\frac{t}{\alpha}+\mathrm{B}\cdot\frac{t^2}{\alpha^2}+\mathrm{C}\cdot\frac{t^3}{\alpha^3}.$$

If now we pursue the reverse process and start with this formula, we shall find the values of A, B, and C, by making t equal to $-\alpha$, α, 2α successively in the equation

$$u_t=a'+\mathrm{A}\cdot\frac{t}{\alpha}+\mathrm{B}\cdot\frac{t^2}{\alpha^2}+\mathrm{C}\cdot\frac{t^3}{\alpha^3}.$$

We thus get

$$\begin{aligned} a&=a'-\mathrm{A}+\mathrm{B}-\mathrm{C},\\ a''&=a'+\mathrm{A}+\mathrm{B}+\mathrm{C},\\ a'''&=a'+2\mathrm{A}+4\mathrm{B}+8\mathrm{C}.\end{aligned}$$

The two first of these equations give

$$a''+a-2a'=2\mathrm{B},$$

whence $\quad \mathrm{B}=\frac{1}{2}\cdot\delta^2 a=\frac{1}{4}(\delta^2 a'+\delta^2 a)-\frac{1}{4}\cdot\delta^3 a.$

Also $\quad a''-a=2(\mathrm{A}+\mathrm{C}).$

But, from the third, $\quad a'''-4\mathrm{B}-a'=2\mathrm{A}+8\mathrm{C};$

whence by subtraction,

$$a'''-a''-(a'-a)-4B=6C,$$

and $$\delta a''-\delta a-2.\delta^2 a=6C;$$

or, since $$\delta a''-\delta a=\delta^2 a'+\delta^2 a=2.\delta^2 a+\delta^3 a,$$

we have $$C=\frac{\delta^3 a}{1.2.3}.$$

Finally, as the second equation gives

$$A=a''-a'-B-C,$$

we shall have $$A=\delta a'-\phi+\frac{1}{12}\cdot\delta^3 a,$$

and substituting these values of A, B, C, we get the formula (n).

It may be noticed that in using the formula (l), if we neglect altogether the last term which involves $\delta^3 a$, the second differences would still be partly corrected by the differences of the third order. In fact, the formula would then give

$$u_t=a'+\frac{t}{\alpha}\cdot\delta a'+\frac{t(t-\alpha)}{\alpha^2}\cdot\phi.$$

But,

$$\phi=\frac{1}{4}(\delta^2 a'+\delta^2 a)=\frac{1}{2}\left(\delta^2 a+\frac{1}{2}\delta^3 a\right)=\frac{1}{2}\left(\delta^2 a'-\frac{1}{2}\delta^3 a\right),$$

so that we shall have

$$u_t=a'+\frac{t}{\alpha}\cdot\delta a'+\frac{t(t-\alpha)}{1.2.\alpha^2}\left(\delta^2 a-\frac{1}{2}\delta^3 a\right).$$

It is from this last term that M. Mathieu has computed a table inserted in the *Connaissance des Temps*. Supposing $\alpha=12^h$, and that the motion of the star is uniform during those 12 hours, the two first terms $a'+\frac{t}{12^h}\cdot\delta a'$ are given; and we then apply the correction resulting from the last term by means of the table in question, which is computed for values of t differing by 10 minutes.

Generally speaking, in all these formulæ, we give to α such integral values as we may require, making it most frequently equal to 12^h, 6^h, 3^h, or 1^h.

On the Value of Apportionable (or Complete) Annuities. (*Continued.*) *By* Thomas B. Sprague, M.A., *Actuary of the Equity and Law Life Assurance Society, and Vice-President of the Institute of Actuaries.*

HAVING thus found the required formula for the value of $\mathbf{a}^{(m)}_{k\overline{|\tau}}$, viz.:—

$$\mathbf{a}^{(m)}_{k\overline{|\tau}}=a_k+\frac{m+1}{2m}-\tau-\frac{\mu+\delta}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)$$
$$-\frac{1}{12m^2}\frac{D'_k}{D_k}\tau(1-m\tau)(1-2m\tau)$$